TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 1, January 1997, Pages 155–169
S 0002-9947(97)01802-3

ON THE ORNSTEIN-UHLENBECK OPERATOR IN L^2 SPACES WITH RESPECT TO INVARIANT MEASURES

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ABSTRACT. We consider a class of elliptic and parabolic differential operators with unbounded coefficients in $\mathbb{R}^n$, and we study the properties of the realization of such operators in suitable weighted L^2 spaces.

1. INTRODUCTION

There is a very wide literature on boundary value problems for linear elliptic and parabolic equations in bounded domains in $\mathbb{R}^n$. A big part of the results can be extended easily to unbounded domains, provided the coefficients of the differential operators are bounded.

A comprehensive approach to the case of unbounded coefficients in $\mathbb{R}^n$ may be found in [1], [2] and [3]. Under appropriate hypotheses, they are able to work in suitably weighted spaces. A typical simple example which is not in general covered by their results is the Ornstein-Uhlenbeck operator

(1.1)
$$\mathcal{A}u = \frac{1}{2}\sum_{i,j=1}^{n} q_{ij}D_{ij}u + \sum_{i,j=1}^{n} b_{ij}x_iD_ju = \frac{1}{2}\operatorname{Tr}(QD^2u) + \langle Bx, Du\rangle, \quad x \in \mathbb{R}^n,$$

and the associated semigroup

(1.2)
$$\begin{cases} (T(t)\varphi)(x) &= \dfrac{1}{(2\pi)^{n/2}(\det\, Q_t)^{1/2}}\displaystyle\int_{\mathbb{R}^n} e^{-\langle Q_t^{-1}y,y\rangle/2}\varphi(e^{tB}x-y)dy, \quad t>0, \\ T(0)\varphi &= \varphi. \end{cases}$$

Here $Q = [q_{ij}]_{i,j=1,\dots,n}$ is any symmetric positive definite matrix, and B is any nonzero matrix. Q_t is the matrix defined by

$$Q_t = \int_0^t e^{sB}Qe^{sB^*}ds, \quad t \ge 0, \tag{1.3}$$

where e^{sB^*} is the exponential of the transpose matrix B^*.

Besides their own mathematical interest, operators with unbounded coefficients arise in stochastic perturbations of ODE's. Consider for instance the linear equation

Received by the editors February 20, 1995.

1991 *Mathematics Subject Classification.* Primary 35J15, Secondary 35K10.

Key words and phrases. Elliptic operators with unbounded coefficients, semigroups, interpolation spaces.

in $\mathbb{R}^n$

$$u'(t) = Bu(t),$$

perturbed by the noise $\sqrt{Q}\,dW_t$, where W_t is a standard n dimensional Brownian motion. Then the semigroup $T(t)$ defined in (1.2) is the Markov semigroup associated to the stochastic differential equation

$$dX = BXdt + \sqrt{Q}\,dW_t,$$

since we have

$$(T(t)\varphi)(x) = \mathbb{E}[\varphi(X(t,x))]$$

for a large class of initial data φ. See e.g. [6, Ch. 5]. It would be obviously important (see again [6]) to study also stochastic perturbations of nonlinear systems

$$u'(t) = Bu(t) + F(u(t)),$$

which give nonlinear coefficients to the operator $\mathcal{A}$. Surprisingly, the literature deals essentially with the case where F is bounded. Our work can be considered as a first step in the study of the more general case of a Lipschitz continuous F. Moreover, at the end of the paper we consider also an example in which $\mathcal{A}$ has variable nonlinear coefficients.

In the paper [4] we have described the properties of the realizations of $\mathcal{A}$ and $T(t)$ in spaces of continuous and bounded functions in $\mathbb{R}^n$. Here we study the realizations of $\mathcal{A}$ and $T(t)$ in a L^2 space with respect to a suitable measure. Besides the usual Lebesgue measure, which will be considered in a forthcoming paper [9], an appropriate measure in the study of a dynamical system is its invariant measure, which exists and is unique under suitable assumptions, see [6]. We assume that all the eigenvalues of the matrix B have negative real part, so that there exist $C > 0$, $\omega > 0$ such that

$$\|e^{tB}\| \le Ce^{-\omega t}, \quad t \in \mathbb{R}. \tag{1.4}$$

Therefore, the matrix

$$Q_\infty = \int_0^\infty e^{sB}Qe^{sB^*}\,ds \tag{1.5}$$

is well defined. We consider the Gaussian weight associated to the matrix Q_∞,

$$\mu(x) = \frac{1}{(2\pi)^{n/2}(\det Q_\infty)^{1/2}} e^{-\langle Q_\infty^{-1}x,x\rangle/2}, \quad x \in \mathbb{R}^n. \tag{1.6}$$

The weighted space L^2_μ is defined by

(1.7)
$$L^2_\mu = \left\{ f : \mathbb{R}^n \mapsto \mathbb{C} \text{ measurable} \,\middle|\, \|f\|_{L^2_\mu} = \left(\int_{\mathbb{R}^n} |f(x)|^2 \mu(x)dx \right)^{1/2} < \infty \right\}.$$

Similarly, the Sobolev weighted space H^s_μ, $s > 0$, is the subspace of L^2_μ consisting of all the functions f such that $x \mapsto f(x)\mu(x)^{1/2}$ belongs to $H^s(\mathbb{R}^n)$. If s is integer, it coincides with the space of all the functions $f \in H^s_{loc}(\mathbb{R}^n)$ such that $D^\beta f$ belongs to L^2_μ for every multi-index β such that $0 \le |\beta| \le s$.

We shall see that the semigroup defined by (1.2) is analytic, strongly continuous, positivity preserving, and it is a contraction semigroup in L^2_μ. Moreover, the measure $\mu(x)dx$ is invariant for $T(t)$, in the sense that

$$\int_{\mathbb{R}^n} (T(t)\varphi)(x)\mu(x)dx = \int_{\mathbb{R}^n} \varphi(x)\mu(x)dx \ \ \forall t \geq 0, \ \varphi \in L^2_\mu. \tag{1.8}$$

The main result of the paper is the characterization of the domain of the realization A of $\mathcal{A}$ in L^2_μ. We prove that $D(A) = H^2_\mu$. The embedding $H^2_\mu \subset D(A)$ is easy, while proving that $D(A) \subset H^2_\mu$ is more delicate. Indeed, due to the strong decay of the weight $\mu(x)$ as $|x| \to \infty$, there are difficulties in treating differential operators in L^2_μ by the usual methods. We use a technique similar to the one employed in [8] to get optimal Schauder type estimates: we show that for every $\alpha \in (0,1)$, $D(A)$ is continuously embedded in the interpolation space

$$(H^\alpha_\mu, H^{2+\alpha}_\mu)_{1-\alpha/2,2} = H^2_\mu.$$

This is done by using the representation formula for the resolvent $R(\lambda, A)$,

$$R(\lambda, A)\varphi = \int_0^\infty e^{-\lambda t} T(t)\varphi dt, \ \ \lambda > 0,$$

and optimal estimates for $\|T(t)\|_{L(L^2_\mu, H^\alpha_\mu)}$, $\|T(t)\|_{L(L^2_\mu, H^{2+\alpha}_\mu)}$, which are obtained with the aid of the explicit representation formula (1.2) and interpolation arguments.

By a similar procedure it is possible to prove that for every $\theta \in (0,1)$ the domain of the realization of $\mathcal{A}$ in H^θ_μ is $H^{2+\theta}_\mu$.

We consider also the case of matrices Q, B depending on x, with continuous coefficients, such that the limits $\lim_{|x|\to\infty} Q(x) = Q$, $\lim_{|x|\to\infty} B(x) = B$ exist, and Q, B satisfy the above assumptions. μ is again the Gaussian weight associated to the matrix Q_∞ defined in (1.5). By using a suitable localization procedure we show that also in this case the domain of the realization A of $\mathcal{A} = \mathrm{Tr}(Q(\cdot)D^2) + \langle B(\cdot)\cdot, D\rangle$ in L^2_μ is H^2_μ, and that A generates an analytic semigroup in L^2_μ.

Once optimal regularity results for elliptic equations have been established, from the general theory of analytic semigroups one gets easily optimal regularity results for parabolic equations,

$$\begin{cases} u_t = \mathcal{A}u + f, \ \ 0 < t < T, \ x \in \mathbb{R}^n, \\ \\ u(0,x) = u_0(x), \ \ x \in \mathbb{R}^n. \end{cases} \tag{1.9}$$

Precisely, one shows that for every $u_0 \in H^1_\mu$ and for every $f \in L^2((0,T) \times \mathbb{R}^n)$ with respect to the measure $dt \times \mu(x)dx$ the solution u of (1.9) is such that u, u_t, $D_i u$, $D_{ij} u$ $(i,j = 1, \dots, n)$ belong to $L^2((0,T) \times \mathbb{R}^n)$ with respect to the measure $dt \times \mu(x)dx$.

2. The spaces L^2_μ and H^s_μ

The space L^2_μ has been defined in (1.7). If there is no danger of confusion we shall write $\|f\|$ instead of $\|f\|_{L^2_\mu}$ for every $f \in L^2_\mu$.

We may assume, possibly changing coordinates, that

$$Q_\infty = \mathrm{diag}(\lambda_1, \dots, \lambda_n),$$

with $\lambda_i > 0$ for $i = 1, \dots, n$. Let Γ be the set of all multi-indexes $\gamma = (\gamma_1, \dots, \gamma_n)$ with $\gamma_i \in \mathbb{N} \cup \{0\}$. For every $\gamma \in \Gamma$ let H_γ be the Hermite polynomial in $\mathbb{R}^n$ associated to the matrix Q_∞, defined by

$$H_\gamma(x) = \prod_{k=1}^{n} H_{\gamma_k}\left(\frac{x_k}{\sqrt{\lambda_k}}\right), \quad x \in \mathbb{R}^n,$$

where for every nonnegative integer r, H_r is the one dimensional r-th Hermite polynomial

$$H_r(x) = \frac{(-1)^r}{\sqrt{r!}}\, e^{x^2/2} \frac{d^r}{dx^r}\, e^{-x^2/2}, \quad x \in \mathbb{R}.$$

It is not hard to see that the set $\{H_\gamma : \gamma \in \Gamma\}$ is an orthonormal basis in L^2_μ.

For $s \in \mathbb{N}$ we define the spaces H^s_μ as

$$H^s_\mu = \{f \in L^2_\mu : \forall |\beta| \le s\ \exists D^\beta f \in L^2_\mu\}, \quad \|f\|_{H^s_\mu} = \sum_{|\beta| \le s} \|D^\beta f\|_{L^2_\mu}.$$

For notational convenience we set also

$$H^0_\mu = L^2_\mu.$$

Using the equalities

$$D_h H_\gamma(x) = \sqrt{\frac{\gamma_h}{\lambda_h}} H_{\gamma_h - 1}\left(\frac{x_h}{\sqrt{\lambda_h}}\right) \prod_{k \ne h} H_{\gamma_k}\left(\frac{x_k}{\sqrt{\lambda_k}}\right), \quad \text{if } \gamma \in \Gamma,\ \gamma_h > 0$$

$$D_h H_\gamma(x) = 0, \quad \text{if } \gamma \in \Gamma,\ \gamma_h = 0,$$

$(h = 1, \dots, n)$ one checks that a function $\varphi \in L^2_\mu$, $\varphi = \sum_{\gamma \in \Gamma} \varphi_\gamma H_\gamma$, is differentiable with respect to x_h with derivative $D_h \varphi \in L^2_\mu$ if and only if

$$\sum_{\gamma \in \Gamma} |\varphi_\gamma|^2 \gamma_h < \infty,$$

in which case we have

$$D_h \varphi(x) = \sum_{\gamma \in \Gamma,\, \gamma_h > 0} \varphi_\gamma \sqrt{\frac{\gamma_h}{\lambda_h}} H_{\gamma_h - 1}\left(\frac{x_h}{\sqrt{\lambda_h}}\right) \prod_{k \ne h} H_{\gamma_k}\left(\frac{x_k}{\sqrt{\lambda_k}}\right),$$

and

$$\|D_h \varphi\|^2_{L^2_\mu} = \frac{1}{\lambda_h} \sum_{\gamma \in \Gamma} |\varphi_\gamma|^2 \gamma_h.$$

Moreover one can see that for every $s \in \mathbb{N}$ a function φ belongs to H^s_μ if and only if

$$\sum_{\gamma \in \Gamma} |\varphi_\gamma|^2 \left(\sum_{h=1}^{n} \gamma_h\right)^s < \infty,$$

and that the norm

$$\varphi \mapsto \left(\sum_{\gamma \in \Gamma} |\varphi_\gamma|^2 \left(\sum_{h=1}^{n} \gamma_h\right)^s\right)^{1/2}$$

is equivalent to the norm of H^s_μ.

A useful property of the space H^1_μ is the following.

Lemma 2.1. *If φ is differentiable with respect to x_h and $D_h\varphi \in L^2_\mu$ then $x \mapsto \psi(x) = x_h\varphi(x)$ belongs to L^2_μ, with norm less than* const. $(\|\varphi\|_{L^2_\mu} + \|D_h\varphi\|_{L^2_\mu})$. *Consequently, if $a : \mathbb{R}^n \mapsto \mathbb{R}$ is any linear function, the mapping $\varphi \mapsto a\varphi$ is bounded from H^1_μ to L^2_μ.*

Proof. It is sufficient to show that for every polynomial φ we have

$$\int_{\mathbb{R}^n} (\psi(x))^2\mu(x)dx \le C(\|\varphi\|^2 + \|D_h\varphi\|^2).$$

If φ is a polynomial, then

$$\begin{aligned}
&\int_{\mathbb{R}^n} (x_h\varphi(x))^2\mu(x)dx = \int_{\mathbb{R}^n} x_h\varphi(x)^2(-\lambda_h/2)(D_h\mu(x))dx \\
&= \frac{\lambda_h}{2}\int_{\mathbb{R}^n} (2x_h\varphi(x)D_h\varphi(x) + \varphi(x)^2)\mu(x)dx \\
&\le \frac{\lambda_h}{2}\|\varphi\|^2 + \lambda_h\left(\int_{\mathbb{R}^n} (x_h\varphi(x))^2\mu(x)dx\right)^{1/2}\|D_h\varphi\| \\
&\le \frac{\lambda_h}{2}\|\varphi\|^2 + \frac{1}{2}\int_{\mathbb{R}^n} (x_h\varphi(x))^2\mu(x)dx + \frac{\lambda_h^2}{2}\|D_h\varphi\|^2.
\end{aligned}$$

Therefore,

$$\|\psi\|^2 \le \lambda_h(\|\varphi\|^2 + \lambda_h\|D_h\varphi\|^2),$$

and the statement follows. □

In the next sections we shall use an explicit characterization of the interpolation spaces $(L^2_\mu, H^s_\mu)_{\theta,2}$. To this aim we define the spaces H^s_μ for $s > 0$ not integer. We set

$$\begin{cases} H^s_\mu = \{f \in L^2_\mu :\ x \mapsto f(x)\exp(-\langle Q_\infty x, x\rangle/4) \in H^s(\mathbb{R}^n)\}, \\[2mm] \|f\|_{H^s_\mu} = \|f\exp(-\langle Q^2_\infty\cdot,\cdot\rangle/4)\|_{H^s(\mathbb{R}^n)}. \end{cases}$$

We define the strongly continuous semigroups $T_h(t)$, $h = 1, \dots, n$, in L^2_μ by

$$T_h(t)\varphi(x) = \varphi(x + te_h)\exp(-(t^2 + 2tx_h)/4\lambda_h),\ \ x \in \mathbb{R}^n,\ t \ge 0.$$

The infinitesimal generator A_h of $T_h(t)$ is the operator defined by

$$\begin{cases} D(A_h) = \{\varphi \in L^2_\mu :\ \exists D_h(\varphi e^{-x_h^2/4\lambda_h}),\ \ x \mapsto e^{x_h^2/4\lambda_h}D_h(\varphi e^{-x_h^2/4\lambda_h}) \in L^2_\mu\}, \\[2mm] A_h\varphi(x) = e^{x_h^2/4\lambda_h}D_h(\varphi(x)e^{-x_h^2/4\lambda_h}) = D_h\varphi - \dfrac{x_h}{2\lambda_h}\varphi. \end{cases}$$

Lemma 2.2. *For every $h = 1, \dots, n$ and $m \in \mathbb{N}$ we have*

$$D(A^m_h) = \{\varphi \in L^2_\mu :\ \exists\partial^m/\partial x^m_h\varphi \in L^2_\mu\},$$

and the graph norm of A^m_h is equivalent to

$$\varphi \mapsto \sum_{k=0}^{m} \|\partial^k/\partial x^k_h\varphi\|_{L^2_\mu}.$$

Proof. Let us prove that the statement holds for $m = 1$. If $\varphi \in L^2_\mu$ is differentiable with respect to x_h with derivative in L^2_μ, by Lemma 2.1 φ belongs to $D(A_h)$. Conversely, if φ is a polynomial then

$$\begin{aligned} A_h\varphi(x) &= \sum_{\tilde\gamma\in\tilde\Gamma}\Big(\sum_{m\geq 0}\varphi_{\tilde\gamma,\,m+1}\sqrt{m+1}(1/\sqrt{\lambda_h}+1/\lambda_h)-\sum_{m\geq 2}\varphi_{\tilde\gamma,\,m-1}\sqrt{m}/\lambda_h\Big)\cdot \\ &\quad \cdot H_m\left(\frac{x_h}{\sqrt{\lambda_h}}\right)\prod_{k=1}^{n-1}H_{\gamma_k}\left(\frac{x_k}{\sqrt{\lambda_k}}\right), \end{aligned}$$

where $\tilde\Gamma$ is the set of all multi-indexes in $(\mathbb{N}\cup\{0\})^{n-1}$ and $\varphi_{\tilde\gamma,\,m+1}$ is the coefficient corresponding to the multi-index $(\tilde\gamma_1,\dots,\tilde\gamma_{h-1},m,\tilde\gamma_h,\dots,\tilde\gamma_{n-1})$. Therefore,

$$\begin{aligned} \|A_h\varphi(x)\|^2 &= \sum_{\tilde\gamma\in\tilde\Gamma}\sum_{m\geq 2}\Big(\varphi_{\tilde\gamma,\,m+1}\sqrt{m+1}(1/\sqrt{\lambda_h}+1/\lambda_h)-\varphi_{\tilde\gamma,\,m-1}\sqrt{m}/\lambda_h\Big)^2 \\ &+(1/\sqrt{\lambda_h}+1/\lambda_h)^2\sum_{\tilde\gamma\in\tilde\Gamma}(\varphi^2_{\tilde\gamma,\,1}+\sqrt{2}\varphi^2_{\tilde\gamma,\,2}) \\ &= \sum_{\tilde\gamma\in\tilde\Gamma}\sum_{m\geq 2}a^2_{\tilde\gamma,\,m}+(1/\sqrt{\lambda_h}+1/\lambda_h)^2\sum_{\tilde\gamma\in\tilde\Gamma}(\varphi^2_{\tilde\gamma,\,1}+\sqrt{2}\varphi^2_{\tilde\gamma,\,2}). \end{aligned}$$

For every $m\geq 2$ and $\varepsilon>0$ we have

$$\begin{aligned} &\Big((1/\sqrt{\lambda_h}+1/\lambda_h)\varphi_{\tilde\gamma,\,m+1}\sqrt{m+1}\Big)^2 = (a_{\tilde\gamma,\,m}+\varphi_{\tilde\gamma,\,m-1}\sqrt{m}/\lambda_h)^2 \\ &\leq (1+1/\varepsilon)a^2_{\tilde\gamma,\,m}+(1+\varepsilon)\varphi^2_{\tilde\gamma,\,m-1}\frac{m}{\lambda_h^2}, \end{aligned}$$

so that for every $M\geq 2$

$$\begin{aligned} &(1/\sqrt{\lambda_h}+1/\lambda_h)^2\sum_{\tilde\gamma\in\tilde\Gamma}\sum_{m\geq M}\varphi^2_{\tilde\gamma,\,m+1}(m+1) \\ &\leq (1+1/\varepsilon)\|A_h\varphi\|^2+C(M)\|\varphi\|^2+(1+\varepsilon)\sum_{\tilde\gamma\in\tilde\Gamma}\sum_{m\geq M}\varphi^2_{\tilde\gamma,\,m-1}\frac{m}{\lambda_h^2} \\ &\leq (1+1/\varepsilon)\|A_h\varphi\|^2+C(M)\|\varphi\|^2 \\ &+(1+\varepsilon)\sum_{\tilde\gamma\in\tilde\Gamma}\lambda_h^{-2}\sum_{m\geq M-2}\varphi^2_{\tilde\gamma,\,m+1}(m+1)\frac{m+2}{m+1}, \end{aligned}$$

which implies

$$\begin{aligned} &\sum_{\tilde\gamma\in\tilde\Gamma}\sum_{m\geq M}\varphi^2_{\tilde\gamma,\,m+1}(m+1) \\ &\leq C(\varepsilon,M)(\|A_h\varphi\|^2+\|\varphi\|^2)+\frac{1+\varepsilon}{(\sqrt{\lambda_h}+1)^2}\;\frac{M+2}{M+1}\sum_{m\geq M-2}\varphi^2_{\tilde\gamma,\,m+1}(m+1). \end{aligned}$$

Taking ε small and M large in such a way that

$$\frac{1+\varepsilon}{(\sqrt{\lambda_h}+1)^2}\ \frac{M+2}{M+1} < 1$$

we get

$$\sum_{\tilde{\gamma}\in\tilde{\Gamma}}\sum_{m\geq 0}\varphi^2_{\tilde{\gamma},\,m+1}(m+1) \leq K(\|A_h\varphi\|^2 + \|\varphi\|^2).$$

Since the set of all polynomials is dense in $D(A_h)$, the statement is proved for $m = 1$. Arguing by recurrence, one can prove that the statement holds for every m. □

The characterization of the domain of A_h^m allows us to characterize the interpolation spaces $(L^2_\mu, H^m_\mu)_{\theta,2}$.

Proposition 2.3. *For every $m \in \mathbb{N}$ and $\theta \in (0,1)$ we have*

$$(L^2_\mu, H^m_\mu)_{\theta,2} = H^{\theta m}_\mu,$$

with equivalence of the respective norms.

Proof. Following [11, §1.13.3] we set

$$K^m = \bigcap_{0\leq k\leq m,\, 1\leq h\leq n} D(A_h^k).$$

By Lemma 2.2 we have $K^m = H^m_\mu$. By [11, Thm. 1.13.6.1] we have

$$(L^2_\mu, H^m_\mu)_{\theta,2} = \{f \in L^2_\mu :\, |||f||| < \infty\},$$

where

$$|||f||| = \|f\|_{L^2_\mu} + \left(\int_{[0,1]^n} |y|^{-2\theta m-n}\left\|\left(\prod_{s=1}^n T_s(y_s) - I\right)^m f\right\|^2_{L^2_\mu} dt\right)^{1/2}.$$

For every $y \in \mathbb{R}^n$ we have

$$\left(\prod_{s=1}^n T_s(y_s) f\right)(x) = f(x+y)e^{-(\langle Q_\infty^{-1}y,y\rangle + 2\langle Q_\infty^{-1}y,x\rangle)/4}, \quad x \in \mathbb{R}^n,$$

so that

$$\left[\left(\prod_{s=1}^n T_s(y_s) - I\right)^m f\right](x)e^{-\langle Q_\infty^{-1}x,x\rangle/4} = \sum_{k=0}^m \binom{m}{k}(-1)^k\varphi(x+ky),$$

where

$$\varphi(x) = f(x)e^{-\langle Q_\infty^{-1}x,x\rangle/4}.$$

By [11, §2.5.1] we get $|||f||| < \infty$ iff $\varphi \in H^{\theta m}(\mathbb{R}^n)$, and the statement follows. □

3. Properties of $T(t)$

The measure $\mu(x)dx$ is invariant for the semigroup $T(t)$, in the sense specified by the following lemma.

Lemma 3.1. *For every $f \in L^1_\mu$ and $t > 0$ we have*

$$\int_{\mathbb{R}^n}(T(t)f)(x)\mu(x)dx = \int_{\mathbb{R}^n} f(x)\mu(x)dx. \tag{3.1}$$

Proof. Since the set of the functions $x \mapsto e^{i\langle h,x\rangle}$ is dense in L^1_μ, it is sufficient to prove that for every $h \in \mathbb{R}^n$ we have

$$\int_{\mathbb{R}^n} (T(t)e^{i\langle h,\cdot\rangle})(x)\mu(x)dx = \int_{\mathbb{R}^n} e^{i\langle h,x\rangle}\mu(x)dx. \tag{3.2}$$

To this aim we remark that

$$(T(t)e^{i\langle h,\cdot\rangle})(x) = e^{i\langle h,e^{tB}x\rangle - \langle Q_t h,h\rangle/2}, \quad t > 0,$$

and

$$\int_{\mathbb{R}^n} e^{i\langle h,x\rangle}\mu(x)dx = e^{-\langle Q_\infty h,h\rangle/2}.$$

So, for $t > 0$ we have

$$\int_{\mathbb{R}^n} (T(t)e^{i\langle h,\cdot\rangle})(x)\mu(x)dx = e^{-(\langle Q_t h,h\rangle + \langle e^{tB}Q_\infty e^{tB^*}h,h\rangle)/2}.$$

Taking into account that

$$Q_t + e^{tB}Q_\infty e^{tB^*} = \int_0^t e^{sB}Qe^{sB^*}ds + \int_0^\infty e^{(t+s)B}Qe^{(t+s)B^*}ds$$

$$= \int_0^t e^{sB}Qe^{sB^*}ds + \int_t^\infty e^{sB}Qe^{sB^*}ds = Q_\infty,$$

(3.2) follows. ☐

Estimates for $T(t)f$ and its derivatives are provided by the following lemma.

Lemma 3.2. *For every $f \in L^2_\mu$ and $t > 0$ we have*

$$\|T(t)f\|_{L^2_\mu} \le \|f\|_{L^2_\mu}, \tag{3.3}$$

$$\|D^\beta T(t)f\|_{L^2_\mu} \le \frac{C}{t^{|\beta|/2}}\|f\|_{L^2_\mu}, \quad |\beta| \le 3. \tag{3.4}$$

Proof. Using the Hölder inequality in formula (1.2) we get

$$|(T(t)f)(x)|^2 \le (T(t)f^2)(x), \quad x \in \mathbb{R}^n,$$

and (3.3) follows from (3.1).

Moreover, setting $\mu_t(y) = (2\pi)^{-n/2}(\det Q_t)^{-1/2}e^{-\langle Q_t^{-1}y,y\rangle/2}$, for every $t > 0$ we have

$$(DT(t)f)(x) = -\int_{\mathbb{R}^n} e^{tB^*}Q_t^{-1}yf(e^{tB}x + y)\mu_t(y)dy.$$

By the Hölder inequality,

$$\|D_iT(t)f\|^2 \le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\langle Q_t^{-1/2}e^{tB}e_i, Q_t^{-1/2}y\rangle|^2\mu_t(y)dy\right.$$

$$\left.\cdot\int_{\mathbb{R}^n}(f(e^{tB}x - y))^2\mu_t(y)dy\right)\mu(x)dx$$

$$\le |Q_t^{-1/2}e^{tB}e_i|^2\int_{\mathbb{R}^n}(T(t)f^2)(x)\mu(x)dx = |Q_t^{-1/2}e^{tB}e_i|^2\|f\|^2_{L^2_\mu} \le Ct^{-1}\|f\|^2_{L^2_\mu},$$

so that (3.4) holds for $|\beta| = 1$.

To estimate the second order derivatives we remark that for every regular $\varphi \in L^2_\mu$ we have

$$DT(t)\varphi = e^{tB^*}T(t)D\varphi, \quad t>0.$$

It follows that for $i,j = 1,\dots,n$ we have

$$\|D_{ij}T(t)f\| = \|D_i(D_jT(t/2)T(t/2)f)\| = \|D_i(e^{tB^*/2}T(t/2)DT(t/2)f)_j\|$$

$$\le C\|DT(t/2)f\|_{(L^2_\mu)^n}\|DT(t/2)\|_{L(L^2_\mu,(L^2_\mu)^n)} \le Ct^{-1}\|f\|,$$

and (3.4) follows for $|\beta| = 2$. The proof for $|\beta| = 3$ is similar. □

From (3.3) and (3.4) it follows by interpolation that for $0<t<1$, $0<\alpha\le 3$

$$\|T(t)\|_{L(L^2_\mu,H^\alpha_\mu)} \le Ct^{-\alpha/2}. \tag{3.5}$$

This estimate is not optimal for t near 0, and it will be improved later. However, we are going to use it in the next proposition to characterize the interpolation spaces $D_A(\theta,2)$, A being the infinitesimal generator of $T(t)$.

It is not difficult to see that the semigroup $T(t)$ is analytic in L^2_μ. In [4, §2.2] one can find a proof which is an adaptation to the finite dimensional case of a result for equations in infinitely many variables due to [7]. Here we give a simple direct proof.

Proposition 3.3. *The semigroup $T(t)$ defined in* (1.2) *is analytic in L^2_μ.*

Proof. By estimates (3.4) with $|\beta| = 2$, for every $f\in L^2_\mu$ and $t>0$ we have

$$\|\sum_{i,j=1}^n q_{ij}D_{ij}T(t)f\|_{L^2_\mu} \le \frac{C}{t}\,\|f\|_{L^2_\mu}.$$

Moreover, due to Lemma 2.1, $x\mapsto \langle Bx, DT(t)f\rangle$ belongs to L^2_μ and by estimates (3.4) we have

$$\|\langle B\cdot, DT(t)f\rangle\|_{L^2_\mu} \le C\|T(t)f\|_{H^1_\mu} \le C(1+t^{-1/2}+t^{-1})\|f\|_{L^2_\mu}.$$

Therefore, for every $f\in L^2_\mu$ and $t>0$

$$\frac{\partial}{\partial t}\,T(t)f = \mathcal{A}T(t)f \in L^2_\mu, \quad \left\|\frac{\partial}{\partial t}\,T(t)f\right\|_{L^2_\mu} \le C(1+t^{-1/2}+t^{-1})\|f\|_{L^2_\mu}.$$

It follows that $t\mapsto T(t)f$ is differentiable for $t>0$ with values in L^2_μ, and

$$\left\|\frac{d}{dt}\,T(t)f\right\|_{L^2_\mu} = \|\mathcal{A}T(t)f\|_{L^2_\mu} \le C(1+t^{-1/2}+t^{-1})\|f\|_{L^2_\mu}, \quad t>0.$$

Since $\|T(t)f\|_{L^2_\mu} \le \|f\|_{L^2_\mu}$ by (3.3), then $T(t)$ is an analytic semigroup. □

We recall that if X is any Banach space and $A : D(A)\subset X\mapsto X$ generates an analytic semigroup $T(t)$ in X, for $0\le\theta<1$ the space $D_A(\theta,2)$ is defined by

$$\begin{cases} D_A(\theta,2) = \{f\in X : \, [f]_{\theta,2} = \displaystyle\int_0^1 t^{1-2\theta}\|AT(t)f\|_X^2 < \infty\}, \\ \\ \|f\|_{D_A(\theta,2)} = \|f\|_X + [f]_{\theta,2}. \end{cases}$$

It is well known that for $\theta\in(0,1)$ the space $D_A(\theta,2)$ coincides with the interpolation space $(X,D(A))_{\theta,2}$, with equivalence of the respective norms. In the next proposition we characterize such spaces.

Proposition 3.4. *We have*

$$D_A(\theta,2)=\begin{cases} L^2_\mu, & \theta=0,\\ \\ H^{2\theta}_\mu, & 0<\theta<1,\end{cases}$$

with equivalence of the respective norms.

Proof. Let $0<\theta<1$. Since H^2_μ is continuously embedded in $D(A)$ by Lemma 2.1, then $H^{2\theta}_\mu=(L^2_\mu,H^2_\mu)_{\theta,2}$ is continuously embedded in $D_A(\theta,2)$. We are going to show that for every $\alpha\in(0,1)$, $H^{2\alpha}_\mu$ belongs to the class J_α between L^2_μ and $D(A)$, i.e.

$$\|f\|_{H^{2\alpha}_\mu}\le C\|f\|^{1-\alpha}_{L^2_\mu}\|f\|^{\alpha}_{D(A)},\quad \forall f\in D(A). \tag{3.6}$$

Indeed, given any $f\in D(A)$, for $\lambda>0$ we have, due to (3.5),

$$\|f\|_{H^{2\alpha}_\mu}=\left\|\int_0^\infty e^{-\lambda t}T(t)(\lambda f-Af)dt\right\|_{H^{2\alpha}_\mu}\le C\int_0^\infty e^{-\lambda t}t^{-\alpha}dt\,\|\lambda f-Af\|_{L^2_\mu}$$

$$\le C'(\lambda^\alpha\|f\|_{L^2_\mu}+\lambda^{\alpha-1}\|Af\|_{L^2_\mu}).$$

Taking the minimum for $\lambda>0$ we get (3.6). Then we may apply the Reiteration Theorem ([11, §1.10]) to get, for $0<\theta<\alpha<1$,

$$D_A(\theta,2)=(L^2_\mu,D(A))_{\theta,2}\subset(L^2_\mu,H^{2\alpha}_\mu)_{\theta/\alpha,2}=H^{2\theta}_\mu,$$

and the statement follows for $0<\theta<1$.

Let now $\theta=0$. We remark preliminarily that if X is a Banach space and A generates a bounded analytic semigroup $T(t)$ in X, then $X=D_A(0,2)$ if and only if $D(A)=(X,D(A^2))_{1/2,2}$. Indeed, we may replace A by $\tilde A=A-I$, $T(t)$ by $\tilde T(t)=T(t)e^{-t}$ and we get $D_A(0,2)=D_{\tilde A}(0,2)$, $(X,D(\tilde A^2))_{1/2,2}=(X,D(A^2))_{1/2,2}$. By [11, §1.14.5],

$$(X,D(\tilde A^2))_{1/2,2}=\{f\in X:\ |||f|||=\int_0^1 t\|\tilde A^2T(t)f\|^2dt<\infty\},$$

and the norm $|||\cdot|||$ is equivalent to the norm of $(X,D(A^2))_{1/2,2}$. Therefore, $(X,D(\tilde A^2))_{1/2,2}=D(\tilde A)$ means that a function f belongs to $D(\tilde A)$ if and only if $|||f|||$ is finite. Since $\tilde A$ is invertible, applying $\tilde A^{-1}$ we get that a function g belongs to X if and only if

$$\int_0^1 t\|\tilde AT(t)g\|^2dt<\infty,$$

which means that $g\in D_{\tilde A}(0,2)$.

So, it is sufficient to prove that $(X,D(A^2))_{1/2,2}=D(A)$. To this aim, we remark that $T(t)$ is a contraction semigroup so that A is m-accretive and it admits bounded imaginary powers (see e.g. [10, §2]). Consequently, by [11, §1.15.3] we get $D(A)=[X,D(A^2)]_{1/2}$ (complex interpolation). Since in our case X and $D(A^2)$ are Hilbert spaces, then $[X,D(A^2)]_{1/2}=(X,D(A^2))_{1/2,2}$ with equivalence of the norms. The statement follows. □

Once the spaces H^α_μ have been characterized as interpolation spaces we may improve estimates (3.5). We shall state just the estimates we need for the sequel. We shall use the following lemma.

Lemma 3.5. *Let A be the generator of an analytic semigroup $T(t)$ in a Banach space X, and let $0 \le \theta < 1$. Then for every $f \in D_A(\theta, 2)$ and $\theta < \alpha < 1$*

$$\int_0^1 t^{2(\alpha-\theta)-1} \|T(t)f\|^2_{D_A(\alpha,2)} dt \le C\|f\|_{D_A(\theta,2)}.$$

Proof. Since $\|T(t)\|_{L(D_A(\theta,2),X)}$ is bounded in $(0,1)$, it is sufficient to prove that $\int_0^1 t^{2(\alpha-\theta)-1}[T(t)f]^2_{D_A(\alpha,2)} dt$ is bounded by $C\|f\|^2_{D_A(\theta,2)}$. Indeed,

$$\int_0^1 t^{2(\alpha-\theta)-1}[T(t)f]^2_{D_A(\alpha,2)} dt = \int_0^1 t^{2(\alpha-\theta)-1} \int_0^1 \xi^{1-2\alpha} \|AT(t+\xi)f\|^2 d\xi$$

$$= \int_0^1 t^{2(\alpha-\theta)-1} \int_t^{t+1} (s-t)^{1-2\alpha} \|AT(s)f\|^2 ds$$

$$\le \int_0^2 \|AT(s)f\|^2 \int_0^s t^{2(\alpha-\theta)-1} (s-t)^{1-2\alpha} dt\, ds$$

$$= \int_0^2 s^{1-2\theta} \|AT(s)f\|^2 \int_0^1 \sigma^{2(\alpha-\theta)-1}(1-\sigma)^{1-2\alpha} d\sigma\, ds$$

$$= C[f]^2_{D_A(\theta,2)}. \quad \square$$

Corollary 3.6. *Let $0 \le \theta < \alpha < 1$, and let $T(t)$ be the semigroup defined in* (1.2). *There exists $C > 0$ such that for every $f \in H^\theta_\mu$ we have*

$$\int_0^1 t^{\alpha-\theta-1} \|T(t)f\|^2_{H^\alpha_\mu} dt \le C\|f\|^2_{H^\theta_\mu}, \tag{3.7}$$

$$\int_0^1 t^{\alpha-\theta+1} \|T(t)f\|^2_{H^{2+\alpha}_\mu} dt \le C\|f\|^2_{H^\theta_\mu}. \tag{3.8}$$

Proof. Estimate (3.7) is an immediate consequence of Proposition 3.4 and Lemma 3.5.

To prove (3.8) we remark that Proposition 3.4 and estimate (3.5) imply that $\|T(t)\|_{L(H^\alpha_\mu, H^{2+\alpha}_\mu)} \le Ct^{-1}$ for $0 < \alpha < 1$, $0 < t < 1$, so that

$$\int_0^1 t^{\alpha-\theta+1} \|T(t)f\|^2_{H^{2+\alpha}_\mu} dt$$

$$\le \int_0^1 t^{\alpha-\theta+1} \|T(t/2)\|^2_{L(H^\alpha_\mu, H^{2+\alpha}_\mu)} \|T(t/2)f\|^2_{H^\alpha_\mu} dt$$

$$\le C\int_0^1 t^{\alpha-\theta-1} \|T(t/2)f\|^2_{H^\alpha_\mu} dt \le C\|f\|^2_{H^\theta_\mu}. \quad \square$$

4. Characterization of the domain of A

The main result of the paper is the following theorem.

Theorem 4.1. *Let $\mathcal{A}$ be the operator defined by* (1.1) *and let A, A_θ be the realizations of $\mathcal{A}$ in L^2_μ and in H^θ_μ, respectively* $(0 < \theta < 1)$. *Then*

$$D(A) = H^2_\mu, \quad D(A_\theta) = H^{2+\theta}_\mu,$$

with equivalence of the respective norms.

Proof. Let us prove the embeddings $\subset$. By Proposition 3.4, $L^2_\mu = D_A(0,2)$ and $H^\theta_\mu = D_A(\theta/2,2)$. So, it is sufficient to show that if $f \in D(A)$ is such that $Af \in D_A(\theta/2,2)$ with $0 \leq \theta < 1$, then $f \in H^{2+\theta}_\mu$. By Proposition 2.3 and the Reiteration Theorem ([11, §1.10]) we have

$$H^{2+\theta}_\mu = (H^\alpha_\mu, H^{2+\alpha}_\mu)_{1-(\alpha-\theta)/2,2}, \ \ \forall \alpha \in (\theta, 1).$$

So, we fix once and for all $\alpha \in (\theta,1)$ and we prove that

$$f \in (H^\alpha_\mu, H^{2+\alpha}_\mu)_{1-(\alpha-\theta)/2,2}. \tag{4.1}$$

Let $\lambda > 0$, and set $\varphi = \lambda f - Af$. Then

$$f = \int_0^\infty e^{-\lambda t} T(t) f \, dt.$$

We recall that if X, Y are Banach spaces such that $Y \subset X$, the interpolation space $(X,Y)_{1-(\alpha-\theta)/2,2}$ is the set of all $f \in X$ such that the function

$$\xi \mapsto k(\xi, f) = \xi^{(-3+\alpha-\theta)/2} \inf_{f=a+b,\, a\in X,\, b \in Y} (\|a\|_X + \xi \|b\|_Y)$$

belongs to $L^2(0,1)$, and the $(X,Y)_{1-(\alpha-\theta)/2,2}$ norm is equivalent to $\|k(\cdot, f)\|_{L^2(0,1)}$. In our case it is convenient to split f as $f = a(\xi) + b(\xi)$, where

$$a(\xi) = \int_0^\xi e^{-\lambda t} T(t) f \, dt, \ \ b(\xi) = \int_\xi^\infty e^{-\lambda t} T(t) f \, dt, \ \ 0 \leq \xi \leq 1.$$

By the Hardy-Young inequality and estimate (3.7) we get

$$\int_0^1 \xi^{\alpha-\theta-3} \|a(\xi)\|^2_{H^\alpha_\mu} d\xi \leq \int_0^1 \xi^{\alpha-\theta-3} \left(\int_0^\xi e^{-\lambda t} \|T(t) f\|_{H^\alpha_\mu} dt \right)^2 d\xi$$

$$\leq \left(1 - \frac{\alpha-\theta}{2}\right)^{-2} \int_0^1 t^{\alpha-\theta-1} \|T(t)f\|^2_{H^\alpha_\mu} dt \leq C \|f\|^2_{D_A(\theta/2,2)}.$$

By the Hardy-Young inequality and estimate (3.8) we get

$$\int_0^1 \xi^{\alpha-\theta-1} \|b(\xi)\|^2_{H^{2+\alpha}_\mu} \int_0^1 \xi^{\alpha-\theta-1} \left(\int_\xi^\infty e^{-\lambda t} \|T(t) f\|_{H^{2+\alpha}_\mu} dt \right)^2 d\xi$$

$$\leq \left(\frac{\alpha-\theta}{2}\right)^{-2} \int_0^\infty t^{\alpha-\theta+1} \|T(t)f\|^2_{H^{2+\alpha}_\mu} dt \leq C \|f\|^2_{D_A(\theta/2,2)}.$$

Since

$$k(\xi, f) \leq \xi^{(-3+\alpha-\theta)/2} (\|a(\xi)\|_{H^\alpha_\mu} + \xi \|b(\xi)\|_{H^{2+\alpha}_\mu}),$$

we get

$$\|k(\cdot, f)\|_{L^2(0,1)} \leq C \|f\|_{D_A(\theta/2,2)},$$

and (4.1) follows.

The embedding $H^2_\mu \subset D(A)$ is an obvious consequence of Lemma 2.1. Moreover, by Lemma 2.1, for every linear function a the mapping $\varphi \mapsto a\varphi$ is bounded from H^1_μ to L^2_μ. Consequently, it is bounded from H^2_μ to H^1_μ. By interpolation, it is bounded from $H^{\theta+1}_\mu$ to H^θ_μ for every $\theta \in (0,1)$. Therefore, $\varphi \mapsto \langle B\cdot, D\varphi \rangle \in L(H^{\theta+2}_\mu, H^\theta_\mu)$, so that $H^{\theta+2}_\mu$ is continuously embedded in $D(A_\theta)$. $\square$

5. The case of coefficients depending on x

Let us consider now the case of coefficients q_{ij}, b_{ij} depending on x. The assumptions on the coefficients are the following:

$$
\left\{\begin{array}{l} q_{ij},\ b_{ij} \in C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n),\ \ q_{ij}=q_{ji},\ i,j=1,\dots,n, \\[2ex] \sum_{i,j=1}^n q_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2\ \ \forall x,\ \xi\in\mathbb{R}^n, \end{array}\right. \tag{5.1}
$$

for some $\nu > 0$. Moreover,

$$
\exists \lim_{|x|\to\infty} q_{ij}(x) = q_{ij}\in\mathbb{R},\ \ \exists \lim_{|x|\to\infty} b_{ij}(x) = b_{ij}\in\mathbb{R}. \tag{5.2}
$$

Setting $Q=[q_{ij}]_{i,j=1,\dots,n}$, $B=[b_{ij}]_{i,j=1,\dots,n}$ we assume that

$$
B \text{ satisfies (1.4).} \tag{5.3}
$$

Let $\mathcal{A}$ be the differential operator defined by

$$
\mathcal{A}f = \frac{1}{2}\sum_{i,j=1}^n q_{ij}(x)D_{ij}f(x) + \sum_{i,j=1}^n b_{ij}(x)x_iD_jf(x),\ \ x\in\mathbb{R}^n. \tag{5.4}
$$

Let μ be the Gaussian weight associated to the matrix Q_∞ defined in (1.5). Define the operators A_∞, $A : H^2_\mu \mapsto L^2_\mu$ by

$$
(A_\infty f)(x) = \mathrm{Tr}(QD^2f(x)) + \langle Bx, Df(x)\rangle,
$$

$$
(Af)(x) = \mathrm{Tr}(Q(x)D^2f(x)) + \langle B(x)x, Df(x)\rangle = \mathcal{A}f(x).
$$

Theorem 5.1. *Under assumptions* (5.1), (5.2), (5.3), $A : D(A) = H^2_\mu \mapsto L^2_\mu$ *generates an analytic semigroup in* L^2_μ.

Proof. As a first step we prove that there are K, $\omega>0$ such that for $\mathrm{Re}\,\lambda > \omega$ and for every $f\in H^2_\mu$

$$
|\lambda|\ \|f\|_{L^2_\mu} + \|f\|_{H^2_\mu} \leq K\|\lambda f - Af\|_{L^2_\mu}. \tag{5.5}
$$

For every $\varepsilon>0$ let $R>1$ be such that

$$
|q_{ij}(x)-q_{ij}| + |b_{ij}(x)-b_{ij}| \leq \varepsilon\ \text{ for } |x|\geq R-1.
$$

Let θ be a smooth cutoff function such that

$$
\left\{\begin{array}{l} \theta\equiv 0\ \text{ in } B(0,R-1),\ \ \theta\equiv 1\ \text{ outside } B(0,R), \\[2ex] |D_i\theta|\leq 1,\ |D_{ij}\theta|\leq 1,\ \ i,j=1,\dots,n. \end{array}\right.
$$

Let $\mathrm{Re}\,\lambda\geq 1$ and let $f\in H^2_\mu$. Then θf satisfies

$$
\begin{aligned}
&\lambda(\theta f)(x) - A_\infty(\theta f)(x) = \theta(x)(\lambda f(x) - Af(x)) \\
&-\theta\Big(\sum_{i,j=1}^n (q_{ij}-q_{ij}(x))D_{ij}f(x) + \sum_{i,j=1}^n (b_{ij}-b_{ij}(x))x_jD_if(x)\Big) \\
&-\sum_{i,j=1}^n q_{ij}(f(x)D_{ij}\theta(x) + 2D_i\theta(x)D_jf(x)) - \sum_{i,j=1}^n b_{ij}x_jf(x)D_i\theta(x)
\end{aligned}
$$

so that by Theorem 4.1 and Proposition 3.3

(5.6)
$$|\lambda|\ \|\theta f\|_{L^2_\mu} + \|\theta f\|_{H^2_\mu} \leq C(\|\theta(\lambda f - Af)\|_{L^2_\mu} + \varepsilon \|f\|_{H^2_\mu} + \|f\|_{H^1_\mu})$$
$$\leq C'(\|\theta(\lambda f - Af)\|_{L^2_\mu} + \varepsilon \|f\|_{H^2_\mu} + C(\varepsilon)\|f\|_{L^2_\mu}).$$

The function $(1-\theta)f$ vanishes outside $B(0,R)$ and satisfies
$$\lambda((1-\theta)f)(x) - A((1-\theta)f)(x) = (1-\theta(x))(\lambda f(x) - Af(x))$$
$$-\sum_{i,j=1}^n q_{ij}(x)(f(x)D_{ij}\theta(x) + D_i\theta(x)D_jf(x))$$
$$-\sum_{i,j=1}^n b_{ij}(x)x_jf(x)D_i\theta(x).$$

By the well known *a priori* estimates for elliptic equations with regular coefficients in bounded sets, if $\operatorname{Re}\lambda$ is large enough we have
$$|\lambda|\ \|(1-\theta)f\|_{L^2(B(0,R))} + \|(1-\theta)f\|_{H^2(B(0,R))}$$
$$\leq C(\|(1-\theta)(\lambda f - Af)\|_{L^2(B(0,R))} + \|f\|_{H^1(B(0,R))} + R\|f\|_{L^2(B(0,R))}),$$

so that for every $\delta > 0$

(5.7)
$$|\lambda|\ \|(1-\theta)f\|_{L^2(B(0,R))} + \|(1-\theta)f\|_{H^2(B(0,R))}$$
$$\leq C(\|(1-\theta)(\lambda f - Af)\|_{L^2(B(0,R))} + \delta\|f\|_{H^2(B(0,R))} + C(\delta,R)\|f\|_{L^2(B(0,R))}).$$

Using (5.6) and (5.7) we get
$$|\lambda|\ \|f\|_{L^2_\mu} + \|f\|_{H^2_\mu} \leq |\lambda|(\|\theta f\|_{L^2_\mu} + C(R)\|(1-\theta)f\|_{L^2(B(0,R))})$$
$$+\|\theta f\|_{H^2_\mu} + C(R)\|(1-\theta)f\|_{H^2(B(0,R))}$$
$$\leq C'\varepsilon\|f\|_{H^2_\mu} + C_1(R)(\|\lambda f - Af\|_{L^2_\mu} + \delta\|f\|_{H^2_\mu} + C(\varepsilon,\delta,R)\|f\|_{L^2(B(0,R))}).$$

Taking ε so small that $C'\varepsilon \leq 1/4$ and then δ so small that $C_1(R)\delta \leq 1/4$ we get
$$|\lambda|\ \|f\|_{L^2_\mu} + \|f\|_{H^2_\mu} \leq \frac{1}{2}\|f\|_{H^2_\mu} + K(\|\lambda f - Af\|_{L^2_\mu} + \|f\|_{L^2_\mu}),$$
and (5.5) follows.

To conclude, we remark that for Re λ large and for every $g \in L^2_\mu$, the equation
$$\lambda f - Af = g$$
has a unique solution $f \in H^2_\mu$. This can be seen using the continuity method: for every $\varepsilon \in [0,1]$ consider the problem

$$\lambda f - (1-\varepsilon)A_\infty f - \varepsilon Af = g. \tag{5.8}$$

Using the *a priori* estimate (5.5) and the fact that A_∞ generates an analytic semigroup it is not hard to see that the set of all ε's such that (5.8) is uniquely solvable in H^2_μ is open and closed in $[0,1]$, so that it coincides with $[0,1]$. Taking $\varepsilon = 1$ the statement follows. □

6. Optimal regularity in parabolic problems

We consider here the parabolic problem

$$\left\{\begin{array}{l} u_t = \mathcal{A}u + f, \quad 0 < t < T, \; x \in \mathbb{R}^n, \\ \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{array}\right. \tag{6.1}$$

where $\mathcal{A}$ is the operator defined in (5.4). We assume that (5.1), (5.2) hold.

We recall that if A is the generator of an analytic semigroup in a Hilbert space H and $f \in L^2(0,T;H)$, $u_0 \in D_A(1/2,2)$, then the problem

$$\left\{\begin{array}{l} u' = Au + f, \quad 0 < t < T, \\ \\ u(0) = u_0, \end{array}\right.$$

has a unique solution $u \in L^2(0,T;D(A)) \cap H^1(0,T;H)$, and

$$\|u\|_{L^2(0,T;D(A))} + \|u\|_{H^1(0,T;H)} \leq C(\|u_0\|_{D_A(1/2,2)} + \|f\|_{L^2(0,T;H)}).$$

Applying this result to problem (6.1) we get that if $u_0 \in H^1_\mu$ and f belongs to $L^2((0,T) \times \mathbb{R}^n)$ with respect to the measure $dt \times \mu(x)dx$ then the solution u of (6.1) is such that u and the derivatives u_t, D_iu, $D_{ij}u$ $(i,j = 1,\dots,n)$ belong to $L^2((0,T) \times \mathbb{R}^n)$ with respect to the measure $dt \times \mu(x)dx$.

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